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Factorization properties of finite spaces

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Abstract
In 1960 Schwinger (J Schwinger 1960 *Proc. Natl Acad. Sci.* **46** 570–9) proposed the algorithm for factorization of unitary operators in the finite M-dimensional Hilbert space according to a coprime decomposition of M. Using a special permutation operator A we generalize the Schwinger factorization to every decomposition of M. We obtain the factorized pairs of unitary operators and show that they obey the same commutation relations as Schwinger's. We apply the new factorization to two problems. First, we show how to generate two kq-like mutually unbiased bases for any composite dimension. Then, using a Harper-like Hamiltonian model in the finite dimension $M = M_1M_2$, we show how to design a physical system with M_1 energy levels, each having degeneracy M_2.

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1. Introduction

A finite phase space of dimension M, where coordinate and momentum have M possible values, is a frequent component of various physical and mathematical problems. Fast Fourier transform (FFT) [1, 2], Schwinger factorization of unitary operators [3], generation of kq bases and finite-dimensional Harper-like Hamiltonians [4] are the problems related to finite phase space that will be considered in this paper. A recent review of various quantum systems with finite Hilbert space can be found in [5].

Originally studied by Weyl [6], the finite-dimensional Hilbert space was systematized by Schwinger in terms of 'unitary operator bases' [3]. Schwinger considered an M-dimensional physical system. Such a Hilbert space can be achieved by application of the following boundary conditions on the wavefunction $\psi(x)$ and its Fourier transform $\Psi(p)$ [7]:

$$\psi(x) = \psi(x + Mc), \qquad \Psi(p) = \Psi\left(p + \frac{2\pi\hbar}{c}\right), \tag{1.1}$$

where c is a length unit. In what follows, we will assume $c = 1$. As a consequence of the above boundary conditions x and p have a finite discrete spectrum of eigenvalues

$$x = 0, 1, \ldots, M-1; \qquad p = \frac{2\pi\hbar}{M} \cdot \{0, 1, \ldots, M-1\}. \tag{1.2}$$

Using unitary operators U and V ([4] with $c = 1$)

$$\begin{cases} U = \mathrm{e}^{\mathrm{i}\hat{x}\frac{2\pi}{M}}, \\ V = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\hat{p}}, \end{cases} \tag{1.3}$$

the complete orthogonal operator basis of M^2 operators can be defined as [3]

$$U^k V^n; \qquad k, n = 0, 1, \ldots, M-1. \tag{1.4}$$

The above operators have the commutation relation

$$V^n U^k = U^k V^n \, \mathrm{e}^{\frac{2\pi\mathrm{i}}{M}nk}. \tag{1.5}$$

For a coprime decomposition of $M = M_1 M_2$, using the Fermat–Euler theorem, Schwinger showed how to factorize the unitary operators. The Fermat–Euler theorem states that if M_1 and M_2 are coprime, then there exist unique N_1 and N_2 such that

$$M_1 N_2 = 1 \ (\mathrm{mod}\ M_2), \qquad M_2 N_1 = 1 \ (\mathrm{mod}\ M_1). \tag{1.6}$$

Therefore, the two pairs of unitary operators defined as

$$\begin{cases} U_1 = U^{M_2}, \\ V_1 = V^{M_2 N_1}, \end{cases} \qquad \begin{cases} U_2 = U^{M_1}, \\ V_2 = V^{M_1 N_2}, \end{cases} \tag{1.7}$$

behave as independent complementary operators of factorized dimensions M_1 and M_2. The respective commutation relations are

$$V_i^{n_i} U_i^{k_i} = U_i^{k_i} V_i^{n_i} \, \mathrm{e}^{\frac{2\pi\mathrm{i}}{M_i}n_i k_i}, \qquad i = 1, 2; \tag{1.8}$$

$$V_i^{n_i} U_j^{k_j} = U_j^{k_j} V_i^{n_i}, \qquad i \neq j, \quad i, j = 1, 2. \tag{1.9}$$

Each unitary operator on the M-dimensional space (equation (1.4)) can be considered as two operators from the factorized dimensions M_1 and M_2. This is due to the one-to-one 'Sino-Ruritanian' correspondences [8]:

$$\begin{aligned} n &= n_1 M_2 N_1 + n_2 M_1 N_2 \ \ (\mathrm{mod}\ M), \\ k &= k_1 M_2 + k_2 M_1 \ \ (\mathrm{mod}\ M). \end{aligned} \tag{1.10}$$

Therefore, for every power n of the operator V we can find the unique representation by the factorized unitary operators V_1 and V_2. The appropriate powers n_1 and n_2 of the factorized operators V_1 and V_2 are determined by the first 'Sino-Ruritanian' correspondence (equation (1.10)). Similarly, the correspondence between U and (U_1, U_2) is determined by the second 'Sino-Ruritanian' correspondence (equation (1.10)). Another recent factorization construction based on the Chinese remainder theorem (CRT) can be found in [5].

After we have obtained factorization of the M-dimensional Hilbert space into its coprime sub-dimensions M_1 and M_2, we can apply it to the kq bases generation and the Harper-like Hamiltonian model. Let us first consider the kq bases generation. The factorized operators from equation (1.7) can be used for generation of the following two pairs of operators (note that $V_2^{M_1} = V^{M_1}$ and $V_1^{M_2} = V^{M_2}$):

$$(a) \begin{cases} \tau\left(\dfrac{2\pi}{a}\right) = \mathrm{e}^{\mathrm{i}\hat{x}\frac{2\pi}{a}} = U^{M_2}; \\ T(a) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\hat{p}a} = V^{M_1}; \end{cases} \qquad (b) \begin{cases} \tau\left(\dfrac{2\pi}{b}\right) = \mathrm{e}^{\mathrm{i}\hat{x}\frac{2\pi}{b}} = U^{M_1}; \\ T(b) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\hat{p}b} = V^{M_2}, \end{cases} \tag{1.11}$$

where the dimension $M = M_1 M_2$ and $a = M_1$, $b = M_2$ (according to the notation of [4] with $c = 1$). Hence, by employing all possible powers, each pair of operators (a) and (b) forms a complete set of M commuting operators and thus generates an alternative kq basis for treatment of the M-dimensional Hilbert space. We have two such bases:

$$(a)\ |k, q\rangle = \frac{1}{\sqrt{M_2}} \sum_{s=0}^{M_2-1} \mathrm{e}^{\mathrm{i}ksa} |q + sa\rangle, \qquad (b)\ |K, Q\rangle = \frac{1}{\sqrt{M_1}} \sum_{t=0}^{M_1-1} \mathrm{e}^{\mathrm{i}Ktb} |Q + tb\rangle$$
$$(a) \begin{cases} k = \dfrac{2\pi}{M} f, \ f = 0, \ldots, M_2 - 1, \\ q = 0, \ldots, M_1 - 1, \end{cases} \qquad (b) \begin{cases} K = \dfrac{2\pi}{M} f', \ f' = 0, \ldots, M_1 - 1, \\ Q = 0, \ldots, M_2 - 1. \end{cases} \tag{1.12}$$

The unique property of the kq bases is that they are eigenfunctions of both space and momentum displacement operators. These functions have partial knowledge about both position and momentum, whose precise simultaneous knowledge is limited by the non-commutation of the corresponding operators. In the case of dimension $M = M_1 M_2$ factorizable to coprime numbers M_1 and M_2, the two kq bases (a) and (b) are mutually unbiased bases (MUB) [4]. The MUB property of the bases means that if the physical system is found in one of the states of one MUB (for example set (a)), then it has equal probabilities to be in all the states of the other MUB (set (b) in our example). Mathematically, the mutual unbiasedness of the two kq bases means the following equality: $|\langle k, q|K, Q\rangle|^2 = \frac{1}{M}$. For non-coprime M_1 and M_2 the MUB property is violated. For example, if $M_1 = m_1 r$ and $M_2 = m_2 r$ have a common multiple r, we have

$$\langle k, q|K, Q\rangle = \frac{1}{\sqrt{M}} \sum_{t,s} \mathrm{e}^{-\mathrm{i}ksa}\, \mathrm{e}^{\mathrm{i}Ktb} \langle sm_1 r + q|tm_2 r + Q\rangle. \tag{1.13}$$

The product $\langle sm_1 r + q|tm_2 r + Q\rangle$ equals unity for the solution of the following modular equation:

$$sm_1 r + q - tm_2 r - Q = 0 \ (\mathrm{mod}\ M). \tag{1.14}$$

Following ([9], p 45, theorem 'd'), the above equation can be taken modulo r:

$$q = Q \ (\mathrm{mod}\ r). \tag{1.15}$$

Therefore, for $q_0 = 0$ and $Q_0 = 1$ (which is always possible according to the ranges of values equation (1.12)) we have $|\langle k, q_0|K, Q_0\rangle|^2 = 0 \neq \frac{1}{M}$.

To complete the introduction to kq MUB we note their quasi-periodic properties:

$$(a)\ \left|k + \frac{2\pi}{M_1}, q\right\rangle = |k, q\rangle, \qquad |k, q + M_1\rangle = \mathrm{e}^{-\mathrm{i}ka} |k, q\rangle,$$
$$(b)\ \left|K + \frac{2\pi}{M_2}, Q\right\rangle = |K, Q\rangle, \qquad |K, Q + M_2\rangle = \mathrm{e}^{-\mathrm{i}Kb} |K, Q\rangle. \tag{1.16}$$

Now, let us consider Harper-like Hamiltonians. They are defined as Hamiltonians of one degree of freedom periodic both in coordinate and momentum [10]. For our discussion we are interested in the use of Harper-like Hamiltonians for the energy spectra design considered in [4]. The energy spectra design is a direct consequence of the factorization of the $M = M_1 M_2$-dimensional Hilbert space to coprime constituents M_1 and M_2. In the original version [4] one considered a Harper-like Hamiltonian $H\left[T(b), \tau\left(\frac{2\pi}{a}\right)\right]$ which is a function of the two operators $T(b)$ and $\tau\left(\frac{2\pi}{a}\right)$ from equation (1.11). It is important that the Hamiltonian $H\left[T(b), \tau\left(\frac{2\pi}{a}\right)\right]$ is a function of the operators V_1 and U_1 (due to $T(b) = V_1^{M_2}$ and $\tau\left(\frac{2\pi}{a}\right) = U_1$); in such a case only the M_1-dimensional subspace is affected by the Hamiltonian. The M_2-dimensional subspace is untouched by the Hamiltonian. Hence, considering $H\left[T(b), \tau\left(\frac{2\pi}{a}\right)\right]$ we expect to

obtain M_1 energy levels (with a spectrum determined by the details of the Hamiltonian) each of which is degenerate M_2 times.

The aim of this paper is first to extend the Schwinger factorization to non-coprime M_1 and M_2. Then the other two applications, the kq-like MUB generation and the energy spectra design by Harper-like Hamiltonian, are extended correspondingly. For that purpose, in section 2, we define the permutation operator A, based on the previous study by Cooley and Tukey of fast Fourier transform (FFT) [2]. Using the operator A we obtain pairs of unitary operators, which have commutation relations as in equations (1.8) and (1.9). In section 3 we use the new factorized unitary pairs to generate two kq-like MUB. New quasi-periodicity properties are obtained in one of the bases. In section 4 we apply the new factorization to the energy spectra design using Harper-like Hamiltonians without any restriction on the factors M_1 and M_2 of the dimension $M = M_1 M_2$. Section 5 includes a discussion and summary.

2. Factorization of unitary operators using the permutation operator A

To define the permutation operator A we start by recalling the division algorithm theorem (DAT) from number theory.

The theorem states ([9] page 2 or [11] page 3) that for any integer numbers D and d with $d > 0$, there exists a unique pair of integer numbers q and r satisfying the following conditions:

$$\begin{aligned} &(a) \quad D = d \cdot q + r, \\ &(b) \quad 0 \leqslant r < d. \end{aligned} \tag{2.1}$$

Consider the special case of positive integer D in the range $[0, 1, \ldots, M_1 M_2 - 1]$ and positive $d = M_2$. In this case there is a unique pair of integers q and r satisfying the following conditions:

$$\begin{aligned} &(a) \quad q \in [0, 1, \ldots, M_1 - 1], \\ &(b) \quad r \in [0, 1, \ldots, M_2 - 1], \\ &(c) \quad D = M_2 \cdot q + r. \end{aligned} \tag{2.2}$$

For our discussion, this DAT-based special representation of the numbers modulo $M = M_1 M_2$ is the crucial component.

In 1965 Cooley and Tukey [2] introduced an FFT algorithm not limited to the coprime factorization of $M = M_1 M_2$. They used two complementary DAT-based representations for the x and p variable indices,

$$\begin{cases} n = n_1 M_2 + n_2, \\ k = k_1 + k_2 M_1, \end{cases} \tag{2.3}$$

which enabled them to simplify the discrete Fourier transform (DFT) calculation ($w_M = \mathrm{e}^{\frac{2\pi i}{M}}$):

$$\begin{aligned} p_k &= \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} x_n w_M^{nk} = \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} x_n w_M^{(n_1 M_2 + n_2)(k_1 + k_2 M_1)} \\ &= \frac{1}{\sqrt{M_2}} \sum_{n_2=0}^{M_2-1} w_M^{n_2(k_1 + k_2 M_1)} \frac{1}{\sqrt{M_1}} \sum_{n_1=0}^{M_1-1} x_n w_M^{n_1 k_1 M_2}. \end{aligned} \tag{2.4}$$

The two summations in the last line of the above formula require $M \cdot \sum_{i=1}^{2} M_i$ operations instead of M^2 operations by direct calculation [2].

Table 1. DAT representation of $x = (x_1, x_2)$.

x	0	1	2	3	4	5
x_1	0	1	0	1	0	1
x_2	0	0	1	1	2	2

Table 2. DAT representation of $x = (x'_1, x'_2)$.

x	0	1	2	3	4	5
x'_1	0	0	0	1	1	1
x'_2	0	1	2	0	1	2

Following Cooley and Tukey, we define a permutation operator A, which acts in the finite M-dimensional Hilbert space:

$$A = \sum_{x_1=0}^{M_1-1} \sum_{x_2=0}^{M_2-1} |x_2 + M_2 x_1\rangle\langle x_1 + M_1 x_2|. \tag{2.5}$$

For the construction of the operator A we used two DAT-based representations, as in equation (2.3), applied to the coordinate states $|x\rangle$. In the coordinate representation our operator is equal to the stride permutation matrix widely used in signal processing [12]. For a simple illustration, let us consider the example of dimension $M = 6$, where $M_1 = 2$ and $M_2 = 3$. Table 1 shows the correspondence between the numbers $x = 0, 1, 2, 3, 4, 5$ and pairs of numbers ($x_1 = 0, 1$; $x_2 = 0, 1, 2$) according to the rule $x = x_1 + M_1 x_2$. With the rule $x = x'_1 M_2 + x'_2$ we have another table (table 2).

Consequently, the permutation matrix A corresponding to equation (2.5) (using the standard basis for $|x\rangle$) is

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.6}$$

Also, it can be written in a compact way as the permutation $A = (0)(1, 3, 4, 2)(5)$. This means that A leaves the coordinate states $|0\rangle$ and $|5\rangle$ unchanged, $|1\rangle$ goes into $|3\rangle$, $|3\rangle$ goes into $|4\rangle$, $|4\rangle$ goes into $|2\rangle$ and $|2\rangle$ goes into $|1\rangle$. We note that the operator A is unitary:

$$AA^\dagger = I. \tag{2.7}$$

With the permutation operator A at hand, we can define the new factorization. To do this we use the pairs of operators from equation (1.11), whose definition includes a general factorization of $M = M_1 M_2$. We modify the (a) set of operators by the permutation operator A of equation (2.5), and for convenience relabel all the operators:

$$(a') \begin{cases} \tilde{U}_1 = \tau'\left(\dfrac{2\pi}{a}\right) = A\tau\left(\dfrac{2\pi}{a}\right)A^\dagger = AU^{M_2}A^\dagger; \\ \tilde{V}_2 = T'(a) = AT(a)A^\dagger = AV^{M_1}A^\dagger; \end{cases} \qquad (b) \begin{cases} \tilde{U}_2 = \tau\left(\dfrac{2\pi}{b}\right) = \mathrm{e}^{\mathrm{i}\hat{x}\frac{2\pi}{b}} = U^{M_1}; \\ \tilde{V}_1 = T(b) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\hat{p}b} = V^{M_2}. \end{cases} \tag{2.8}$$

The tilde denotes the new version of operators. The (a') and (b) sets of operators replace the Schwinger operators of equation (1.7). As we will show shortly, they obey all the commutation relations (equations (1.8) and (1.9)) of factorized operators. Therefore, the (a') and (b) sets of operators define the new factorization, not restricted to coprime decomposition. The commutation relation (1.9) is fulfilled by the tilde operators (equation (2.8)) due to the unitarity property of A. For the commutation relation (1.8) we first calculate the operation of $AU^{M_2}A^\dagger$ and $A^\dagger U^{M_1}A$ on coordinate states:

$$\begin{aligned} AU^{M_2}A^\dagger|x\rangle &= AU^{M_2}A^\dagger|x_1M_2+x_2\rangle = AU^{M_2}|x_1+M_1x_2\rangle \\ &= \mathrm{e}^{\frac{2\pi\mathrm{i}}{M_1}x_1}A|x_1+M_1x_2\rangle = \mathrm{e}^{\frac{2\pi\mathrm{i}}{M_1}x_1}|x_1M_2+x_2\rangle; \end{aligned} \tag{2.9}$$

$$\begin{aligned} A^\dagger U^{M_1}A|x\rangle &= A^\dagger U^{M_1}A|x_1'+M_1x_2'\rangle = A^\dagger U^{M_1}|x_1'M_2+x_2'\rangle \\ &= \mathrm{e}^{\frac{2\pi\mathrm{i}}{M_2}x_2'}A^\dagger|x_1'M_2+x_2'\rangle = \mathrm{e}^{\frac{2\pi\mathrm{i}}{M_2}x_2'}|x_1'+M_1x_2'\rangle, \end{aligned} \tag{2.10}$$

where the only difference in the above calculations is that we used different DAT-based representations for the x values. Using the above expressions one can prove

$$\begin{aligned} \tilde{V}_1^{n_1}\tilde{U}_1^{k_1}|x\rangle &= V^{M_2n_1}AU^{M_2k_1}A^\dagger|x_1M_2+x_2\rangle = \mathrm{e}^{\frac{2\pi\mathrm{i}}{M_1}k_1x_1}V^{M_2n_1}|x_1M_2+x_2\rangle \\ &= \mathrm{e}^{\frac{2\pi\mathrm{i}}{M_1}k_1x_1}|(x_1-n_1)M_2+x_2\rangle, \end{aligned} \tag{2.11}$$

whereas applying $\tilde{U}_1^{k_1}\tilde{V}_1^{n_1}$ we get

$$\begin{aligned} \tilde{U}_1^{k_1}\tilde{V}_1^{n_1}|x\rangle &= AU^{M_2k_1}A^\dagger V^{M_2n_1}|x_1M_2+x_2\rangle = AU^{M_2k_1}A^\dagger|(x_1-n_1)M_2+x_2\rangle \\ &= \mathrm{e}^{\frac{2\pi\mathrm{i}}{M_1}k_1(x_1-n_1)}|(x_1-n_1)M_2+x_2\rangle. \end{aligned} \tag{2.12}$$

Similar results can be shown for the operators $\tilde{V}_2^{n_2}$ and $\tilde{U}_2^{k_2}$. Summarizing the results, the commutation relation (equation (1.8)) is fulfilled:

$$\tilde{V}_i^{n_i}\tilde{U}_i^{k_i} = \tilde{U}_i^{k_i}\tilde{V}_i^{n_i}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{M_i}n_ik_i}, \qquad i=1,2. \tag{2.13}$$

Therefore, using the permutation operator A we obtained the new factorization of the unitary operators, which is not limited to coprime decomposition of M. Here we used the permutation operator A for the transformation of the (a) set of operators to the new (a') set for the factorization. Obviously we could have applied the transformation to the (b) set, which would have also enabled the factorization.

In the particular case of $M_1 = M_2$ $(a = b)$, the two kq bases (1.12) are identical, and so are the two (a) and (b) sets of operators in equation (1.11), and the permutation operator A from equation (2.5) satisfies $A^2 = I$. In this case we have only one set of kq operators and only one $|k,q\rangle$ basis. Application of the permutation operator A to that set of operators defines the tilde set, which obeys the proper commutation relations with the original set (equations (1.8) and (1.9)). Their respective eigenstates (obtained by applying A to the unique $|k,q\rangle$ basis) are MUB with respect to the original $|k,q\rangle$ states. (See also the next example and the treatment of Harper-like Hamiltonians for $M = 4 = 2^2$ in section 4.)

The permutation operator A, based on the analogy to the Cooley and Tukey FFT, solves the unitary operator factorization. To acquire some physical intuition about the operator A, let us consider the example of dimension $M = 4$, where $M_1 = M_2 = 2$. In this case, the operators from equation (1.11) in the coordinate representation may be presented as

$$(a)\left\{U^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad V^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \right. \tag{2.14}$$

The sets (a) and (b) of operators in equation (1.11) are identical in our example. To get the second set of factorized operators we write the (a') set from equation (2.8):

$$(a')\begin{cases} AU^2A^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad AV^2A^\dagger = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \end{cases} \tag{2.15}$$

where the permutation operator in the coordinate representation is given by the permutation $A = (0)(1,2)(3)$. The operator U^2 has two eigenvalues (1 and -1), and the operator V^2 permutes between the vectors with the same eigenvalue (1 or -1). This is why the operator U^2 commutes with the operator V^2. Permutation by the operator A turns the operator U^2 into the operator $AU^2A^\dagger$, which anticommutes with V^2. The operators $AU^2A^\dagger$ and V^2 form a complementary pair of operators for sub-dimension $M_1 = 2$, where their anticommutation is consistent with equation (1.8). The operators U^2 and $AV^2A^\dagger$ form another complementary pair of operators for sub-dimension $M_2 = 2$. The operator A permutes the eigenvalues of U^2 in such a way as to make the operator V^2 anticommute with $AU^2A^\dagger$.

A more interesting example is dimension $M = 12$, where both coprime and non-coprime factorizations are possible. For the case of $M_1 = 2$ and $M_2 = 6$, using the coordinate representation, the permutation operator is

$$A = (0)(1,6,3,7,9,10,5,8,4,2)(11).$$

In the other case, where $M_1 = 3$ and $M_2 = 4$, the permutation operator is

$$A = (0)(1,4,5,9,3)(2,8,10,7,6)(11).$$

In both cases the construction of the operators from equation (2.8) leads to the factorized pairs of operators. The generality of the new factorization enables us to perform it for every factorized numbers M_1 and M_2. In the case where M_1 or M_2 are composite numbers, another factorization can be performed until we reach prime numbers in factorization.

Note that Schwinger's solution for non-coprime factorization in [3] gives the factorized pairs of operators (see also [13]), which obey the commutation relations of equations (1.8) and (1.9). However, while for the coprime factorization an explicit expression is given in [3] connecting between the factorized pairs and the original operators U and V, no such expression is given for the non-coprime case. In our paper this explicit expression is given in equation (2.8).

3. New *kq*-like bases

Each set (a') and (b) of operators (equation (2.8)) generates M commuting operators and can be used for the definition of a basis for the M-dimensional Hilbert space. The set (b) of operators has, as an eigenbasis, the $|K, Q\rangle$ basis. As a result of the unitary transformation of the (a) set, the (a') set defines the kq-like basis $\widetilde{|k, q\rangle}$ as follows:

$$\begin{aligned} &(a')\ \widetilde{|k,q\rangle} = A|k,q\rangle = \frac{1}{\sqrt{M_2}} \sum_{s=0}^{M_2-1} \mathrm{e}^{\mathrm{i}ksa}|s + qM_2\rangle, \\ &(a')\begin{cases} k = \dfrac{2\pi}{M} f, & f = 0, \ldots, M_2 - 1, \\ q = 0, \ldots, M_1 - 1. \end{cases} \end{aligned} \tag{3.1}$$

As a result of the fact that the two sets of operators $(\tilde{V}_1, \tilde{U}_1)$ and $(\tilde{V}_2, \tilde{U}_2)$ describe M_1 and M_2 subspaces in the entire M-dimensional Hilbert space, the bases $|\widetilde{k, q}\rangle$ and $|K, Q\rangle$ are mutually unbiased. Let us check the overlap between $|\widetilde{k, q}\rangle$ and $|K, Q\rangle$ states ($a = M_1,\ b = M_2$):

$$\langle\widetilde{k, q}|K, Q\rangle = \frac{1}{\sqrt{M_1}}\frac{1}{\sqrt{M_2}}\sum_{s=0}^{M_2-1}\sum_{t=0}^{M_1-1} \mathrm{e}^{-\mathrm{i}ksa}\,\mathrm{e}^{\mathrm{i}Ktb}\langle s + qM_2|Q + tb\rangle. \tag{3.2}$$

Inserting ($a = M_1, b = M_2$) we have

$$\begin{aligned}
&= \frac{1}{\sqrt{M}}\sum_{s=0}^{M_2-1}\sum_{t=0}^{M_1-1} \mathrm{e}^{-\mathrm{i}ksM_1}\,\mathrm{e}^{\mathrm{i}KtM_2}\langle s + qM_2|Q + tM_2\rangle \\
&= \frac{1}{\sqrt{M}}\sum_{s=0}^{M_2-1}\sum_{t=0}^{M_1-1} \mathrm{e}^{-\mathrm{i}ksM_1}\,\mathrm{e}^{\mathrm{i}KtM_2}\delta^{M_1}(s - Q)\delta^{M_2}(q - t) \\
&= \frac{1}{\sqrt{M}}\,\mathrm{e}^{-\mathrm{i}kQM_1}\,\mathrm{e}^{\mathrm{i}KqM_2}.
\end{aligned} \tag{3.3}$$

Here $\delta^{M_i}(x - x_0)$ means that the argument of the delta function is taken modulo M_i, $\delta^{M_i}(0) = 1$ and elsewhere is zero. Therefore, these bases are mutually unbiased: $|\langle\widetilde{k, q}|K, Q\rangle|^2 = \frac{1}{M}$. We call the basis $|\widetilde{k, q}\rangle$ a kq-like basis because it is not an eigenfunction of the same operators as the $|k, q\rangle$ basis, but of the operators related to them by the permutation transformation. In addition, it has different periodicity properties. As $|\widetilde{k, q}\rangle$ is defined, it has the completely periodic property:

$$(a')\left|\widetilde{k + \frac{2\pi}{M_1}, q}\right\rangle = |\widetilde{k, q + M_1}\rangle = |\widetilde{k, q}\rangle. \tag{3.4}$$

To show explicitly the difference and similarity between the bases $|\widetilde{k, q}\rangle$ and $|k, q\rangle$ we consider an example of dimension $M = 6$ with $M_1 = 3$ and $M_2 = 2$. Using the $|x\rangle$ representation we list in three columns all basis members of $|k, q\rangle$ on the left-hand side, all $|\widetilde{k, q}\rangle$ basis vectors in the middle and $|K, Q\rangle$ on the right-hand side:

$$\begin{cases}
|(0,0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |3\rangle), \\
|(0,1)\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |4\rangle), \\
|(0,2)\rangle = \frac{1}{\sqrt{2}}(|2\rangle + |5\rangle), \\
|(1,0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |3\rangle), \\
|(1,1)\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |4\rangle), \\
|(1,2)\rangle = \frac{1}{\sqrt{2}}(|2\rangle - |5\rangle),
\end{cases}
\qquad
\begin{cases}
|\widetilde{(0,0)}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \\
|\widetilde{(0,1)}\rangle = \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle), \\
|\widetilde{(0,2)}\rangle = \frac{1}{\sqrt{2}}(|4\rangle + |5\rangle), \\
|\widetilde{(1,0)}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), \\
|\widetilde{(1,1)}\rangle = \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle), \\
|\widetilde{(1,2)}\rangle = \frac{1}{\sqrt{2}}(|4\rangle - |5\rangle),
\end{cases}$$

$$\begin{cases}
|(0,0)\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |2\rangle + |4\rangle), \\
|(0,1)\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |3\rangle + |5\rangle), \\
|(1,0)\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}|2\rangle + \mathrm{e}^{\frac{4\pi\mathrm{i}}{3}}|4\rangle\right), \\
|(1,1)\rangle = \frac{1}{\sqrt{3}}\left(|1\rangle + \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}|3\rangle + \mathrm{e}^{\frac{4\pi\mathrm{i}}{3}}|5\rangle\right), \\
|(2,0)\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + \mathrm{e}^{\frac{4\pi\mathrm{i}}{3}}|2\rangle + \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}|4\rangle\right), \\
|(2,1)\rangle = \frac{1}{\sqrt{3}}\left(|1\rangle + \mathrm{e}^{\frac{4\pi\mathrm{i}}{3}}|3\rangle + \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}|5\rangle\right).
\end{cases} \tag{3.5}$$

Hence, the $|\widetilde{k, q}\rangle$ and $|k, q\rangle$ bases are neither equal nor orthogonal to one another (they are eigenfunctions of different sets of operators). Nevertheless, both these bases are mutually unbiased to the $|K, Q\rangle$ basis in this coprime case.

4. Engineering of the energy spectrum using the permutation operator A in Harper-like Hamiltonians

The new factorized pairs of operators $\left(T'(a), \tau\left(\frac{2\pi}{b}\right)\right)$ and $\left(T(b), \tau'\left(\frac{2\pi}{a}\right)\right)$ (equation (2.8)) describe M_2- and M_1-dimensional subspaces, respectively [3]. Therefore, replacing the Harper-like Hamiltonian $H\left[T(b), \tau\left(\frac{2\pi}{a}\right)\right]$ of [4] by $H\left[T(b), \tau'\left(\frac{2\pi}{a}\right)\right]$ we should obtain M_1 energy levels, each of which is degenerate M_2 times, without restriction for M_2 and M_1 to be coprime.

To show the advantage of the new factorization, we compare it with the energy spectra design method of [4]. As a first example, let us consider the dimension $M = 6$. We choose the simple Harper-like Hamiltonian proposed in [4]:

$$H = H\left(T(b), \tau\left(\frac{2\pi}{a}\right)\right) = V_1 \cos\left(\frac{b}{\hbar}\hat{p}\right) + V_2 \cos\left(\frac{2\pi}{a}\hat{x}\right), \tag{4.1}$$

where V_1 and V_2 are constants. We solve this Hamiltonian using the kq-representation:

$$|\psi\rangle = \sum_{k,q} |k, q\rangle\langle k, q|\psi\rangle = \sum_{k,q} C_{k,q}|k, q\rangle. \tag{4.2}$$

The resulting eigenvalue equation for our Hamiltonian is

$$\left[V_1 \cos\left(\frac{b}{\hbar}\hat{p}\right) + V_2 \cos\left(\frac{2\pi}{a}\hat{x}\right)\right] \sum_{k,q} C_{k,q}|k, q\rangle = \varepsilon \sum_{k,q} C_{k,q}|k, q\rangle. \tag{4.3}$$

After applying the operators we have

$$\sum_{k,q} C_{k,q}\left[\frac{V_1}{2}(|k, q - b\rangle + |k, q + b\rangle) + V_2 \cos\left(\frac{2\pi}{a}q\right)|k, q\rangle\right] = \varepsilon \sum_{k,q} C_{k,q}|k, q\rangle. \tag{4.4}$$

The above eigenvalue equation can be solved for each value of $k = \frac{2\pi}{M}f$ independently. So for our particular choice of dimension $M = 6$ with $M_1 = a = 2$ and $M_2 = b = 3$, performing the summation over q values with the use of the quasi-periodicity property of the $|k, q\rangle$ states, we obtain the following equation (for some particular k value):

$$\begin{aligned} C_{k,0}\left[\frac{V_1}{2}(\mathrm{e}^{4ki}|k, 1\rangle + \mathrm{e}^{-2ki}|k, 1\rangle) + V_2 \cos\left(\frac{2\pi}{2}\cdot 0\right)|k, 0\rangle\right] \\ + C_{k,1}\left[\frac{V_1}{2}(\mathrm{e}^{2ki}|k, 0\rangle + \mathrm{e}^{-4ki}|k, 0\rangle) + V_2 \cos\left(\frac{2\pi}{2}\cdot 1\right)|k, 1\rangle\right] \\ = \varepsilon[C_{k,0}|k, 0\rangle + C_{k,1}|k, 1\rangle]. \end{aligned} \tag{4.5}$$

Using the orthogonality of the $|k, q\rangle$ states the above equation is equivalent to the solution of the following $M_1 = 2$ coupled equations:

$$\begin{pmatrix} V_2 & V_1 \mathrm{e}^{-\frac{2\pi i}{6}4f} \\ V_1 \mathrm{e}^{\frac{2\pi i}{6}4f} & -V_2 \end{pmatrix} \begin{pmatrix} C_{k,0} \\ C_{k,1} \end{pmatrix} = \varepsilon \begin{pmatrix} C_{k,0} \\ C_{k,1} \end{pmatrix}. \tag{4.6}$$

The energy spectrum ε is

$$\varepsilon_{1,2} = \pm\sqrt{V_1^2 + V_2^2}, \tag{4.7}$$

which is f independent and therefore each energy level is threefold degenerate (note that for current example $f = \{0, 1, 2\}$ and $k = \frac{2\pi}{6} f$). The relation between the coefficients is

$$C_{k,0} = \frac{V_1 \, \mathrm{e}^{-\frac{2\pi \mathrm{i}}{6} 4f}}{\varepsilon - V_2} C_{k,1} \qquad \text{or equally} \qquad C_{k,1} = \frac{V_1 \, \mathrm{e}^{\frac{2\pi \mathrm{i}}{6} 4f}}{\varepsilon + V_2} C_{k,0}. \tag{4.8}$$

On the other hand, if instead of coprime factorized $M = 6$ we choose $M = 4$ and substitute $M_1 = a = 2$ and $M_2 = b = 2$ into equation (4.4), we get equation (4.9) with non-degenerate energy spectrum:

$$\sum_{k,q} C_{k,q} \left[\frac{V_1}{2} (|k, q-2\rangle + |k, q+2\rangle) + V_2 \cos\left(\frac{2\pi}{2} q\right) |k, q\rangle \right] = \varepsilon \sum_{k,q} C_{k,q} |k, q\rangle. \tag{4.9}$$

Using the quasi-periodicity properties of $|k, q\rangle$ states we have

$$\sum_{k,q} C_{k,q} [V_1 \cos(2k) |k, q\rangle + V_2 \cos\left(\frac{2\pi}{2} q\right) |k, q\rangle] = \varepsilon \sum_{k,q} C_{k,q} |k, q\rangle, \tag{4.10}$$

and the energy spectrum is

$$\varepsilon_{1,2,3,4} = \pm V_1 \pm V_2. \tag{4.11}$$

This result is expected, because of the absence of factorization into sub-dimensions $M_1 = 2$ and $M_2 = 2$ using the operators of equation (1.11).

Let us now follow the same procedure with the new operators of equation (2.8). Accordingly, the Harper-like Hamiltonian of equation (4.1) changes to

$$H = H\left(T(b), \tau'\left(\frac{2\pi}{a}\right)\right) = V_1 \cos\left(\frac{b}{\hbar}\hat{p}\right) + V_2 A \cos\left(\frac{2\pi}{a}\hat{x}\right) A^{\dagger}. \tag{4.12}$$

To compare the two schemes we solve the above Hamiltonian using the $\widetilde{kq}$-representation:

$$|\psi\rangle = \sum_{k,q} |\widetilde{k, q}\rangle \langle \widetilde{k, q} | \psi\rangle = \sum_{k,q} \widetilde{C}_{k,q} |\widetilde{k, q}\rangle. \tag{4.13}$$

The eigenvalue equation for our Hamiltonian is

$$\left[V_1 \cos\left(\frac{b}{\hbar}\hat{p}\right) + V_2 A \cos\left(\frac{2\pi}{a}\hat{x}\right) A^{\dagger} \right] \sum_{k,q} \widetilde{C}_{k,q} |\widetilde{k, q}\rangle = \varepsilon \sum_{k,q} \widetilde{C}_{k,q} |\widetilde{k, q}\rangle. \tag{4.14}$$

After applying the operators we have

$$\sum_{k,q} \widetilde{C}_{k,q} \left[\frac{V_1}{2} (|\widetilde{k, q-1}\rangle + |\widetilde{k, q+1}\rangle) + V_2 \cos\left(\frac{2\pi}{a} q\right) |\widetilde{k, q}\rangle \right] = \varepsilon \sum_{k,q} \widetilde{C}_{k,q} |\widetilde{k, q}\rangle, \tag{4.15}$$

where we have used the two relations:

$$T(b) |\widetilde{k, q}\rangle = |\widetilde{k, q-1}\rangle \qquad \text{and} \qquad \tau'\left(\frac{2\pi}{a}\right) |\widetilde{k, q}\rangle = \mathrm{e}^{\frac{2\pi \mathrm{i}}{a} q} |\widetilde{k, q}\rangle.$$

As before, the eigenvalue equation (4.15) can be solved for each value of k independently, and using the complete periodicity property of $|\widetilde{k, q}\rangle$ we have (with $M_1 = a = 2$ and $M_2 = b = 3$)

$$\begin{aligned} \tilde{C}_{k,0} &\left[\frac{V_1}{2} (|\widetilde{k, 1}\rangle + |\widetilde{k, 1}\rangle) + V_2 \cos\left(\frac{2\pi}{2} \cdot 0\right) |\widetilde{k, 0}\rangle \right] \\ &+ \tilde{C}_{k,1} \left[\frac{V_1}{2} (|\widetilde{k, 0}\rangle + |\widetilde{k, 0}\rangle) + V_2 \cos\left(\frac{2\pi}{2} \cdot 1\right) |\widetilde{k, 1}\rangle \right] \\ &= \varepsilon [\tilde{C}_{k,0} |\widetilde{k, 0}\rangle + \tilde{C}_{k,1} |\widetilde{k, 1}\rangle]. \end{aligned} \tag{4.16}$$

In matrix form the above equation reads

$$\begin{pmatrix} V_2 & V_1 \\ V_1 & -V_2 \end{pmatrix} \begin{pmatrix} \tilde{C}_{k,0} \\ \tilde{C}_{k,1} \end{pmatrix} = \varepsilon \begin{pmatrix} \tilde{C}_{k,0} \\ \tilde{C}_{k,1} \end{pmatrix}. \tag{4.17}$$

Hence, we get the same spectrum of energies as before, with each level being threefold degenerate:

$$\varepsilon_{1,2} = \pm\sqrt{V_1^2 + V_2^2}, \tag{4.18}$$

and a new relation between the coefficients

$$\tilde{C}_{k,0} = \frac{V_1}{\varepsilon - V_2}\tilde{C}_{k,1} \qquad \text{or equally} \qquad \tilde{C}_{k,1} = \frac{V_1}{\varepsilon + V_2}\tilde{C}_{k,0}. \tag{4.19}$$

In the case of $M = 4$, solving equation (4.15) with $M_1 = a = 2$ and $M_2 = b = 2$, we have

$$\sum_{k,q} \widetilde{C}_{k,q} \left[\frac{V_1}{2}(|\widetilde{k, q-1}\rangle + |\widetilde{k, q+1}\rangle) + V_2 \cos\left(\frac{2\pi}{2}q\right) |\widetilde{k, q}\rangle \right] = \varepsilon \sum_{k,q} \widetilde{C}_{k,q} |\widetilde{k, q}\rangle. \tag{4.20}$$

As a result of the k independence of the equation above it is equivalent to the eigenvalue equation considered for dimension $M = 6$. Therefore, (as one can easily check) we have to solve the matrix equation

$$\begin{pmatrix} V_2 & V_1 \\ V_1 & -V_2 \end{pmatrix} \begin{pmatrix} \tilde{C}_{k,0} \\ \tilde{C}_{k,1} \end{pmatrix} = \varepsilon \begin{pmatrix} \tilde{C}_{k,0} \\ \tilde{C}_{k,1} \end{pmatrix}, \tag{4.21}$$

and consequently the corresponding energy levels, with each level being twofold degenerate, are

$$\varepsilon_{1,2} = \pm\sqrt{V_1^2 + V_2^2}. \tag{4.22}$$

Therefore, in spite of the non-degenerate spectrum of the Hamiltonian $H\big(T(b), \tau\big(\frac{2\pi}{a}\big)\big)$ for non-coprime M_1 and M_2, for the Hamiltonian $H\big(T(b), \tau'\big(\frac{2\pi}{a}\big)\big)$ the energy levels preserve their degeneracies.

5. Summary and discussion

The main result of our work is a generalization of the Schwinger unitary operator factorization to non-coprime factorizations. That is, for a composite dimension $M = M_1 M_2$, we factorize the U and V operators from equation (1.3) into two pairs of operators $(\tilde{U}_1, \tilde{V}_1)$ and $(\tilde{U}_2, \tilde{V}_2)$ (2.8). Each of the pairs generates a complete orthogonal operator basis for the sub-dimensions M_1 and M_2, and operators from different bases commute. The factorization enables us to consider any single physical system with dimension $M = M_1 M_2$ as a pair of physical systems in M_1- and M_2-factorized degrees of freedom, where M_1 and M_2 are not restricted to be coprime. Considering factorized operators may simplify various M-dimensional phase space problems in the same way as the Cooley–Tukey FFT simplifies the application of the DFT. Moreover, the new factorization deepens our physical intuition. In particular, we applied the new factorization to a Harper-like Hamiltonian model, and developed an algorithm for energy spectrum design in this model. Using the algorithm, we can construct a Hamiltonian, which is a function of the operators $(\tilde{U}_1, \tilde{V}_1)$. Therefore, it is designed to obtain M_1 energy levels (with a spectrum determined by Hamiltonian's details), each level being M_2-fold degenerate. The algorithm of the energy spectrum design can be of interest, for example, in solid state physics for electrons in a strong magnetic field [14].

The application of the permutation operator A (which is the key to the solution for the non-coprime cases) to the kq bases problem generates the kq-like basis which is a MUB to the original $|K, Q\rangle$ basis. This kq-like basis has a different periodicity property than the original kq bases: it is completely periodic in the coordinate and momentum variables simultaneously.

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